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The rook problem on saw-toothed chessboards

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Abstract

A saw-toothed chessboard, or STC for short, is a kind of chessboard whose boundary forms two staircases from left down to right without any hole inside it. A rook at square (i, j) can dominate the squares in row i and in column j. The rook problem of an STC is to determine the minimum number of rooks that can dominate all squares of the STC. In this paper, we model an STC by two particular graphs: a rook graph and a board graph. We show that for an STC, the rook graph is the line graph of the board graph, and the board graph is a bipartite permutation graph. Thus, the rook problem on STCs can be solved by any algorithm for solving the edge domination problem on bipartite permutation graphs.

Keywords: Rook problem; Chessboard; Dominating set; Bipartite permutation graph; Algorithm

1. Introduction

A *chessboard* is an $m \times n$ array of squares, where m is the number of rows and n is the number of columns. A *hole* is a missing square in a chessboard. If a chessboard has holes, it may be in some particular shape. A *saw-toothed chessboard*, or STC for short, is a kind of chessboard whose boundary forms two staircases from left down to right without any hole inside it. Fig. 1 shows two examples of chessboards: Fig. 1(a) is a chessboard with two holes, while Fig. 1(b) is a saw-toothed chessboard.

On an $m \times n$ chessboard, we can put one *rook* at some square (i, j), where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then, this rook can dominate the squares in row i and in column j. The *rook problem* of a chessboard is to determine the minimum number of rooks that can dominate all squares of the chessboard. For example, in Fig. 1(b), all squares can be dominated by four rooks located at squares (1, 1), (3, 2), (4, 6), and (5, 3).

The chessboard domination problem was first mentioned in 1862 [4] which can also be found in [2]. One of the well-known chessboard domination problems is the n-queens problem. The *n-queens problem* is to place n queens on an $n \times n$ chessboard so that no two queens threaten each other; i.e. no two queens may be in the same row, column, or diagonal. In 1892, Ball gave the values of the minimum dominating number of the queen domination problem on

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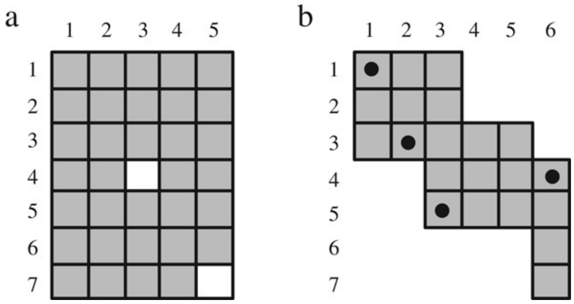

Fig. 1. Examples of chessboards. (a) A 7 × 5 chessboard with two holes. (b) A 7 × 6 saw-toothed chessboard.

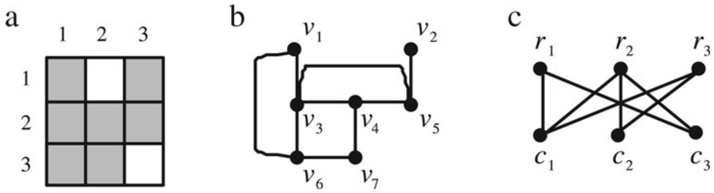

Fig. 2. A chessboard and its rook graph and board graph. (a) A chessboard. (b) The corresponding rook graph. (c) The corresponding board graph.

an $n \times n$ board, where $n \leq 8$ [1]. The chessboard domination problem is still studied, and some recent results can be found in [2,3,9].

The rook problem is different to the n-queens problem. A rook can move to any position in the row or column in which it lies, while a queen can move to any position in the row, column, or diagonals in which it lies [6]. In [11], Yaglom et al. showed that the minimum number of rooks to dominate an $n \times n$ chessboard is n.

In this paper, we discuss some properties of the rook problem on STCs. We model an STC by two particular graphs: a rook graph and a board graph. We show that for an STC, the rook graph is the line graph of the board graph, and the board graph is a bipartite permutation graph. Then, we conclude that the rook problem on STCs can be solved by any algorithm for finding a minimum edge dominating set on bipartite permutation graphs.

2. Some properties of the rook problem on STCs

In this section, we introduce some properties of the rook problem on STCs. An undirected graph G_r is called the *rook graph* of a chessboard if each vertex of G_r corresponds to a distinct square of the chessboard such that two vertices of G_r are adjacent if and only if their corresponding squares are in the same row or in the same column without any hole between them. For example, Fig. 2(b) shows the rook graph of the chessboard in Fig. 2(a), where vertices v_1, v_2, v_3, v_4, v_5, v_6, and v_7 correspond to squares (1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), and (3, 2), respectively. Since square (1, 2) is a hole in the chessboard, there is no edge incident to both v_1 and v_2. Moreover, vertices v_1, v_3, and v_6 form a clique since there is no hole in column 1.

The *board graph* G_b of a chessboard has two parts of vertices, say r vertices and c vertices, such that row i (respectively, column j) of the chessboard corresponds to vertex r_i (respectively, c_j) of G_b and there is an edge incident to vertices r_i and c_j in G_b if and only if square (i, j) is not a hole. Fig. 2(c) shows the board graph of the chessboard in Fig. 2(a).

An $m \times n$ saw-toothed chessboard can be represented by m tuples. A tuple (r_i, c_i, d_i) describes the squares in row r_i of the STC, where c_i (respectively, d_i) is the column index of the first (respectively, the last) non-hole square in row r_i. Notice that $c_1 = 1$ and $d_m = n$. Thus, the STC in Fig. 1(b) can be represented by tuples (1, 1, 3), (2, 1, 3), (3, 1, 5), (4, 3, 6), (5, 3, 6), (6, 6, 6), and (7, 6, 6). By the definition of an STC, we have the following property.

Property 1. *Let (r_i, c_i, d_i) and (r_j, c_j, d_j) be two tuples of an $m \times n$ STC, where $1 \leq i < j \leq m$, then $c_i \leq c_j$ and $d_i \leq d_j$. Moreover, $c_j \leq d_i + 1$ if $j = i + 1$.*

Let G_r and G_b be the rook graph and board graph respectively of an $m \times n$ STC. Then, we have the following lemma.

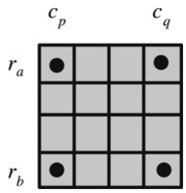

Fig. 3. G_b has a strong ordering of $X \cup Y$.

Lemma 2. *Let x and y be two vertices of G_r, and let e_x and e_y be two edges of G_b corresponding to x and y. Then, (x, y) is an edge of G_r if and only if e_x and e_y are adjacent in G_b.*

Proof. By the definition of a rook graph, if (x, y) is an edge of G_r, then the squares of the STC corresponding to x and y are in the same row or in the same column. This implies that e_x and e_y are incident to a common vertex in G_b. For the other part of this proof, since e_x and e_y are adjacent, the squares of the STC corresponding to e_x and e_y are in the same row or in the same column. Since these two squares are not holes, by the definition of an STC, there is no hole between them. Therefore, there is an edge incident to both x and y in G_r. □

Let G be a graph. The *line graph* $L(G)$ of G is defined as follows [5]: Each vertex of $L(G)$ corresponds to a distinct edge in G and two vertices of $L(G)$ are adjacent if and only if the two corresponding edges in G are adjacent. For any line graph $L(G)$, we say G is the *original graph* of $L(G)$.

Lemma 3. *G_r is the line graph of G_b.*

Proof. It is from the definition of a board graph and Lemma 2. □

A graph G is a *bipartite graph* if its vertex set can be partitioned into two subsets X and Y such that every edge of G joins X with Y [5]. In a bipartite graph $G = (X \cup Y, E)$, a *strong ordering* of the vertices of G consists of an ordering of X and an ordering of Y such that for all (x, y') and (x', y) in E, where $x, x' \in X$ and $y, y' \in Y$, if $x < x'$ and $y < y'$, then (x, y) and (x', y') are in E. A bipartite graph $G = (X \cup Y, E)$ is a *bipartite permutation graph* if there exists a strong ordering of $X \cup Y$ [7,10].

Lemma 4. *G_b is a bipartite permutation graph.*

Proof. Clearly, G_b is a bipartite graph, and we let $G_b = (X \cup Y, E)$. Without loss of generality, assume that X (respectively, Y) has m (respectively, n) vertices in the ordering of $r_1, r_2, \ldots, r_m$ (respectively, $c_1, c_2, \ldots, c_n$), in which r_i (respectively, c_j) corresponds to row i (respectively, column j) of the STC. Let r_a, r_b be two vertices of X and c_p, c_q be two vertices of Y, where $r_a < r_b$ and $c_p < c_q$. Suppose that (r_a, c_q) and (r_b, c_p) are two edges of G_b. Then, squares (r_a, c_q) and (r_b, c_p) of the STC are not holes. Since square (r_b, c_p) is not a hole, square (r_a, c_p) cannot be a hole by Property 1. Similarly, square (r_a, c_q) is not a hole, and square (r_b, c_q) cannot be a hole. (See Fig. 3.) Thus, (r_a, c_p) and (r_b, c_q) are two edges of G_b, and G_b is a bipartite graph with a strong ordering of $X \cup Y$. □

Let $G = (V, E)$ be a graph. A *vertex dominating set* D_V of G is a subset of V such that every vertex of G not in D_V is adjacent to one vertex of D_V. A *minimum vertex dominating set* has the minimum number of vertices among all vertex dominating sets. Similarly, an *edge dominating set* D_E of G is a subset of E such that every edge of G not in D_E is adjacent to one edge of D_E, and a *minimum edge dominating set* is an edge dominating set with the minimum number of edges. Then, we have the following result.

Theorem 5. *The following problems are equivalent:*
(i) *finding the minimum number of rooks of an $m \times n$ STC,*
(ii) *finding a minimum vertex dominating set of G_r,*
(iii) *finding a minimum edge dominating set of G_b.*

Proof. We complete this proof by two equivalences.
(i) is equivalent to (ii): It is directly from the definition of a rook graph.
(ii) is equivalent to (iii): Let G be a graph and $L(G)$ its line graph. Clearly, an edge dominating set of G corresponds to a vertex dominating set of $L(G)$, and vice versa. Thus, the problem of finding a minimum vertex dominating set of G_r is equivalent to the problem of finding a minimum edge dominating set of G_b. □

3. Concluding remarks

In this paper, we define a saw-toothed chessboard (or STC) and model it by its rook graph G_r and board graph G_b. We also introduce some properties of the rook problem on STCs. Finally, we have the result that the rook problem of an STC is equivalent to the problem of finding a minimum edge dominating set of G_b. Since G_b is a bipartite permutation graph, we can solve the rook problem of an STC by applying any algorithm for finding a minimum edge dominating set on bipartite permutation graphs. In [8], Srinivasan et al. proposed an $O(nm + n^2)$ time algorithm for the edge domination problem on bipartite permutation graphs, where n is the number of vertices and m the number of edges. If we use their algorithm, the rook problem of an $m \times n$ STC can be solved in $O((m + n)mn)$ time since the number of vertices in G_b is $m + n$ and the number of edges in G_b is at most mn. We are trying to find a more efficient algorithm for the rook problem on STCs.

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